

GEOMETRIC-TYPE SOBOLEV INEQUALITIES AND APPLICATIONS TO THE REGULARITY OF MINIMIZERS

XAVIER CABRÉ AND MANEL SANCHÓN

ABSTRACT. The purpose of this paper is twofold. We first prove a weighted Sobolev inequality and part of a weighted Morrey's inequality, where the weights are a power of the mean curvature of the level sets of the function appearing in the inequalities. Then, as main application of our inequalities, we establish new L^q and $W^{1,q}$ estimates for semi-stable solutions of $-\Delta u = g(u)$ in a bounded domain Ω of \mathbb{R}^n . These estimates lead to an $L^{2n/(n-4)}(\Omega)$ bound for the extremal solution of $-\Delta u = \lambda f(u)$ when $n \geq 5$ and the domain is convex. We recall that extremal solutions are known to be bounded in convex domains if $n \leq 4$, and that their boundedness is expected—but still unknown—for $n \leq 9$.

1. INTRODUCTION

The main purpose of this paper is twofold. On the one hand, we prove the following geometric-type Sobolev and Morrey's inequalities for functions $v \in C_0^\infty(\bar{\Omega})$, where Ω is a smooth bounded domain of \mathbb{R}^n with $n \geq 2$. Assume that $p \geq 1$ and $r \in \{0\} \cup [1, \infty)$. Then, there exists a constant C depending only on n , p , and r , such that the following inequalities hold for all $v \in C_0^\infty(\bar{\Omega})$:

$$\|v\|_{L^{p_r^*}(\Omega)} \leq C \left\| |H_v|^r |\nabla v| \right\|_{L^p(\Omega \cap \{|\nabla v| > 0\})} \quad \text{if } n > p(1+r)$$

and

$$\|v\|_{L^\infty(\Omega)} \leq C |\Omega|^{\frac{p(1+r)-n}{np}} \left\| |H_v|^r |\nabla v| \right\|_{L^p(\Omega \cap \{|\nabla v| > 0\})} \quad \text{if } 1+r \leq n < p(1+r).$$

Here, the critical exponent p_r^* is defined by

$$\frac{1}{p_r^*} := \frac{1}{p} - \frac{1+r}{n}$$

and the function H_v appearing in the right hand side of both inequalities denotes the mean curvature of the level sets of $|v|$ (which are smooth hypersurfaces at points where $|\nabla v| > 0$). In particular, it depends on v in a nonlinear way, given by the expression

$$H_v = \frac{-1}{n-1} \operatorname{div} \left(\frac{\nabla v}{|\nabla v|} \right).$$

We also establish a related inequality of Trudinger type when $n = p(1+r)$.

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On the other hand, as an application of these inequalities, we derive new L^q and $W^{1,q}$ *a priori* estimates for local minimizers (and more generally, for semi-stable solutions) of reaction-diffusion problems. These estimates motivated the study of the geometric Sobolev inequalities above.

Consider the reaction-diffusion problem

$$(1.1) \quad \begin{cases} -\Delta u &= g(u) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases}$$

where g is any C^1 function. We say that a classical solution $u \in C^2(\overline{\Omega})$ of (1.1) is *semi-stable* if

$$(1.2) \quad \int_{\Omega} \{|\nabla \xi|^2 - g'(u)\xi^2\} dx \geq 0 \quad \text{for all } \xi \in C_0^1(\overline{\Omega}).$$

This class of solutions includes local minimizers of the associated energy functional, minimal solutions, extremal solutions, and also certain solutions found between a sub and a super solution. We use the semi-stability condition (1.2) with the test function $\xi = |\nabla u|\eta$. Using this choice of ξ and an equation for $(\Delta + g'(u))|\nabla u|$, one deduces that

$$(1.3) \quad (n-1) \int_{\Omega \cap \{|\nabla u| > 0\}} H_u^2 |\nabla u|^2 \eta^2 dx \leq \int_{\Omega} |\nabla u|^2 |\nabla \eta|^2 dx$$

for every Lipschitz function η in $\overline{\Omega}$ with $\eta|_{\partial\Omega} \equiv 0$. We take $\eta \equiv 1$ in a compact set $K \subset \Omega$, and thus $|\nabla \eta|$ is supported in $\overline{\Omega} \setminus K$. Then, if we know that u is regular in a neighborhood of $\partial\Omega$ (this holds for instance when Ω is convex) and we take K big enough, the right hand side of (1.3) is bounded. We deduce that

$$\int_{K \cap \{|\nabla u| > 0\}} H_u^2 |\nabla u|^2 dx \leq C$$

and, with the help of our Sobolev inequality above with $r = 1$ and $p = 2$, we establish a new bound:

$$u \in L^{2n/(n-4)}(\Omega) \quad \text{if } n \geq 5 \text{ and } \Omega \text{ is convex.}$$

Moreover, using this $L^{2n/(n-4)}$ estimate, we are also able to obtain $W^{1,q}$ bounds for semi-stable solutions. This result completes the L^∞ estimate obtained by the first author in [6] whenever $n \leq 4$ and Ω is convex.

For general domains and increasing positive and convex nonlinearities g , Nedev [17] proved an L^∞ bound when $n \leq 3$, and an L^q estimate, for every $q < n/(n-4)$ when $n \geq 4$. Note that the exponent $2n/(n-4)$ in our L^q bound above improves the one of Nedev. Besides, we make no assumption on the nonlinearity, but in contrast with Nedev's result, we assume Ω to be convex.

2. MAIN RESULTS

2.1. Geometric-type Sobolev inequalities. We start stating the Sobolev and Morrey's type inequalities involving the mean curvature of the level sets.

Theorem 2.1. *Let Ω be a smooth bounded domain of \mathbb{R}^n , with $n \geq 2$. Let $p \geq 1$ and $r \in \{0\} \cup [1, \infty)$.*

Let $v \in C_0^\infty(\overline{\Omega})$ (i.e., $v \in C^\infty(\overline{\Omega})$ and $v = 0$ on $\partial\Omega$). For $x \in \Omega$ with $\nabla v(x) \neq 0$, let $H_v(x)$ be the mean curvature at x of the hypersurface $\{y \in \Omega : |v(y)| = |v(x)|\}$, which is smooth at x . The following assertions hold:

(a) Assume either that $1 + r \leq n < p(1 + r)$ or that $n = 1 + r$ and $p = 1$. Then

$$(2.1) \quad \|v\|_{L^\infty(\Omega)} \leq C_1 |\Omega|^{\frac{p(1+r)-n}{np}} \left(\int_{\Omega \cap \{|\nabla v| > 0\}} |H_v|^{pr} |\nabla v|^p dx \right)^{1/p},$$

for some constant C_1 depending only on n , p , and r .

(b) If $n > p(1 + r)$, then

$$(2.2) \quad \left(\int_{\Omega} |v|^{p_r^*} dx \right)^{1/p_r^*} \leq C_2 \left(\int_{\Omega \cap \{|\nabla v| > 0\}} |H_v|^{pr} |\nabla v|^p dx \right)^{1/p},$$

where $\frac{1}{p_r^*} := \frac{1}{p} - \frac{1+r}{n}$, for some constant C_2 depending only on n , p , and r .

(c) If $p > 1$ and $n = p(1 + r)$, then

$$(2.3) \quad \int_{\Omega} \exp \left\{ \left(\frac{|v|}{C_3 \left(\int_{\Omega \cap \{|\nabla v| > 0\}} |H_v|^{pr} |\nabla v|^p dx \right)^{1/p}} \right)^{p'} \right\} dx \leq C_4 |\Omega|,$$

where $p' = p/(p-1)$, and C_3 and C_4 are positive constants depending only on n and p .

In Remarks 3.1 and 3.3 we give explicit expressions for admissible values of the constants C_i , $i = 1, \dots, 4$, in the theorem. These expressions involve two isoperimetric constants A_1 and A_2 (only A_1 when $r = 0$) that we describe next.

Note that Theorem 2.1 is well known for $r = 0$. Indeed, (a) states a part of Morrey's inequality, (b) is the classical Sobolev inequality, and (c) is Trudinger's inequality. It is well known that they follow from the classical isoperimetric inequality, which states that for any smooth bounded domain D of \mathbb{R}^n ,

$$(2.4) \quad A_1 |D|^{(n-1)/n} \leq |\partial D|$$

where $A_1 = n|B_1|^{1/n}$ and B_1 denotes the unit ball in \mathbb{R}^n . Our proof will show this fact and that admissible constants in the theorem are completely explicit in terms only of A_1 , n , and p when $r = 0$.

To establish the theorem when $r \geq 1$ we need another isoperimetric inequality. It involves the mean curvature H of C^2 immersed $(n-1)$ -dimensional compact hypersurfaces without boundary $S \subset \mathbb{R}^n$, and states

$$(2.5) \quad |S|^{\frac{n-2}{n-1}} \leq A_2 \int_S |H(x)| d\sigma.$$

Here, H is the mean curvature of S , $d\sigma$ denotes the area element in S , and A_2 is a universal constant depending only on the dimension $n \geq 2$. When $n = 2$, (2.5) follows from the Gauss-Bonnet formula. When $n \geq 3$, the inequality is due to Michael and Simon [16] and to Allard [1] —see Theorem 28.4.1 [5] for a more general version of (2.5). From such a version, a Sobolev inequality for functions defined on hypersurfaces S of \mathbb{R}^n , and which involves the mean curvature H of S , can be deduced (see section 28.5 of [5], Theorem 2.1 [6], or Theorem C.2.1 [9]).

Remark 2.2 (The critical exponents). Note that the critical exponent p_r^* in part (b) of the theorem coincides with the classical Sobolev exponent in the embedding $W^{1+r,p} \subset L^{p_r^*}$ for functions with $1 + r$ derivatives in L^p .

The critical case in Theorem 2.1 corresponds to $n = p(1 + r)$. It is given in part (a) when $p = 1$ and in part (c) when $p > 1$. In the second case, $p > 1$, the L^∞ estimate does not necessarily hold, as usual. This can be easily seen using radial functions when Ω is a ball. Instead, the embedding in L^∞ holds in the critical case when $p = 1$ (and thus $n = 1 + r$), as in the classical case $W^{n,1} \subset L^\infty$.

Note that in all cases of Theorem 2.1 we have $1 + r \leq n$. In the case $p = 1$ and $n < 1 + r$, which is not covered by Theorem 2.1, we derive an inequality involving the total variation of $|v|$ in Remark 3.5.

Remark 2.3 (The case $p = +\infty$). Letting p tend to $+\infty$ in (2.1) and using the explicit constant C_1 obtained in Remark 3.1, we deduce

$$(2.6) \quad \|v\|_{L^\infty(\Omega)} \leq \frac{n}{1+r} \left(n|B_1|^{1/n} \right)^{\frac{1+r-n}{n-1}} A_2^r |\Omega|^{\frac{1+r}{n}} \| |H_v|^r |\nabla v| \|_{L^\infty(\Omega \cap \{|\nabla v| > 0\})}$$

when $n \geq 2$ and $1 \leq r \leq n - 1$. Here, A_2 is a constant depending only on the dimension n for which (2.5) holds.

Remark 2.4 (The case $r \in (0, 1)$). Theorem 2.1 is stated for $r = 0$ and $r \geq 1$. A natural question is if it does not hold for $r \in (0, 1)$ independently of the dimension n . In this direction, in Remark 3.4 we prove that Theorem 2.1 (a)-(b) do not hold for $r \in (0, 2p^{-1} - 1)$ when $1 \leq p < 2$, independently of the dimension. In particular, they do not hold for $r \in (0, 1)$ when $p = 1$.

For the class of mean convex functions—that is, functions whose level sets have nonnegative mean curvature—the estimates in Theorem 2.1 can be established in the larger range $r \geq 1/p$. The argument only applies to mean convex functions since it relies on the fact that the perimeter of the level sets of a mean convex function v is a nonincreasing function, *i.e.*, $|\{x \in \Omega : |v(x)| = t_1\}| \geq |\{x \in \Omega : |v(x)| = t_2\}|$ for a.e. $0 < t_1 < t_2$. When $r = 1/p$, such estimates were proven by Trudinger [21]. The inequalities in [21] carry optimal constants and are claimed there to hold for all mean convex functions. However, at present they are only known to hold for functions with *starshaped* and mean convex level sets. The reason is that to obtain optimal constants one needs to use inequality (2.5) with the constant A_2 which makes (2.5) to be an equality when S is a sphere. That such constant A_2 is admissible in (2.5) is still only known among *starshaped* mean convex hypersurfaces S , by a recent result of Guan and Li [13]; see also [14].

Theorem 2.1 can be used to study the geometric flow of mean convex hypersurfaces driven by a positive power r of their mean curvature, the so-called H^r -flow. The theorem leads, for instance, to upper bounds on the extinction time of the flow. In the level set formulation, the flow can be represented by the level sets of a mean convex function v satisfying the elliptic equation

$$H_v = \frac{-1}{n-1} \operatorname{div} \left(\frac{\nabla v}{|\nabla v|} \right) = \frac{1}{|\nabla v|^{1/r}}.$$

Noting that $\| |H_v|^r |\nabla v| \|_{L^\infty(\Omega \cap \{|\nabla v| > 0\})} = 1$ and using (2.6) one obtains an L^∞ estimate for v , or equivalently, an upper bound for the extinction time of the H^r -flow. Let us mention here that Schulze [19] used the H^r -flow to give a new proof of a deep result of B. Kleiner: the Euclidean isoperimetric inequality also holds for domains of any complete and simply-connected 3-dimensional manifold with nonpositive sectional curvatures—a result that is still open for the same type of manifolds of dimension $n \geq 5$.

In this respect, Theorem 2.1 could be extended to the case of functions defined on Riemannian manifolds. Indeed, the first ingredient in our proof —the coarea formula— holds on any Riemannian manifold. On the other hand, the isoperimetric inequalities that we use to prove the theorem could be replaced by those in the particular manifold; see section 36.5 of [5].

Remark 2.5 (The radial case). When $\Omega = B_R = B_R(0)$, if we restrict inequality (2.2) to radially symmetric functions v with compact support in B_R then (2.2) reads

$$(2.7) \quad \left(\int_0^R |v(\rho)|^q \rho^{n-1} d\rho \right)^{1/q} \leq C \left(\int_0^R \rho^{-pr} |v'(\rho)|^p \rho^{n-1} d\rho \right)^{1/p},$$

where $q = p_r^*$. Here $\rho = |x|$. Note that in the radial case, the level set at x , $\{|v| = |v(x)|\}$, is a sphere of radius $|x|$, and thus the average of its principal curvatures is $H_v(x) = |x|^{-1} = \rho^{-1}$. The 1-dimensional weighted Sobolev inequality (2.7) has been well studied (see [15] for this one and more general versions). It is well known that (2.7) holds, with a constant C independent of v , if and only if either $n < p(1+r)$ and $q \leq +\infty$, or $n > p(1+r)$ and $q \leq p_r^*$, or $n = p(1+r)$ and $q < +\infty$. This shows that Theorem 2.1 (b) is sharp in terms of the exponents that it involves and the restrictions on them. The sharpness in this same sense of parts (a) and (c) of Theorem 2.1 can also be checked using radially decreasing functions.

Remark 2.6 (Relation with a Caffarelli-Kohn-Nirenberg inequality). Since $H_v(x) = |x|^{-1}$ for radial functions, Theorem 2.1 (b) is related to the Caffarelli, Kohn, and Nirenberg inequality [8], which states the following. *Assume $q > 0$, $p \geq 1$, and $n > pr$. Then, there exists a positive constant C such that*

$$(2.8) \quad \|v\|_{L^q(\mathbb{R}^n)} \leq C \| |x|^{-r} |\nabla v| \|_{L^p(\mathbb{R}^n)}$$

holds for all $v \in C_0^\infty(\mathbb{R}^n)$, if and only if $q = p_r^$ and $-1 \leq r \leq 0$.* Here, the condition $r \leq 0$ is due to the unboundedness of the domain and to the fact that the singularity of the weight is fixed at the origin —and thus (2.8) is not invariant under translations. Indeed, that $r \leq 0$ is necessary in (2.8) can be shown by taking $v(x) = u(x - x_0)$ with $u \in C_0^\infty(B_1)$ and letting $|x_0| \rightarrow +\infty$.

Instead, our inequalities are invariant under translations.

The second part of this paper is devoted to obtain, as an application of Theorem 2.1, *a priori* estimates for semi-stable solutions of the reaction-diffusion problem (1.1) —which motivated the present work.

2.2. Application to the regularity of stable solutions and extremal solutions. Applying Theorem 2.1 we obtain *a priori* estimates for semi-stable solutions of (1.1). In particular, for the extremal solution u^* of $(2.13)_\lambda$ below —*i.e.*, problem (1.1) when $g(u) = \lambda f(u)$.

Recently, the first author proved the boundedness of the extremal solution of $(2.13)_\lambda$ when the domain is convex and $n \leq 4$. Our following result is the main application of Theorem 2.1. We establish an $L^{\frac{2n}{n-4}}$ estimate for the extremal solution in convex domains when $n \geq 5$. For these domains, the result improves the L^q for $q < n/(n-4)$ and the $L^{\frac{2n}{n-2}}$ estimates of Nedev proved, respectively, in [17] and [18].

Theorem 2.7. *Let $f : [0, +\infty) \rightarrow \mathbb{R}$ be an increasing positive C^1 function (in particular with $f(0) > 0$) such that $f(t)/t \rightarrow +\infty$ as $t \rightarrow +\infty$. Assume that Ω is a convex smooth bounded domain of \mathbb{R}^n with $n \geq 5$. Let u^* be the extremal solution of $(2.13)_\lambda$. Then,*

$$u^* \in L^{\frac{2n}{n-4}}(\Omega).$$

The convexity assumption on the domain Ω is only used to control the L^∞ -norm of u^* in a neighborhood of the boundary $\partial\Omega$. For general domains and general nonlinearities we are able to prove the following *a priori* estimates for semi-stable solutions —from which Theorem 2.7 will follow easily.

Theorem 2.8. *Let g be any C^∞ function and $\Omega \subset \mathbb{R}^n$ any smooth bounded domain with $n \geq 5$. Let $u \in C_0^1(\overline{\Omega})$ be a semi-stable solution of (1.1), i.e., a solution satisfying (1.2). Then,*

$$(2.9) \quad \left(\int_{\{|u|>s\}} (|u| - s)^{\frac{2n}{n-4}} dx \right)^{\frac{n-4}{2n}} \leq \frac{C(n)}{s} \left(\int_{\{|u|\leq s\}} |\nabla u|^4 dx \right)^{1/2}$$

for all $s > 0$, where $C(n)$ is a constant depending only on n . Moreover,

$$(2.10) \quad \int_{\Omega} |\nabla u|^p dx \leq p|\Omega| + \left(\frac{4n}{(3n-4)p} - 1 \right)^{-1} \left\{ \int_{\Omega} |u|^{\frac{2n}{n-4}} dx + \|g(u)\|_{L^1(\Omega)} \right\}$$

for all $1 \leq p < \frac{4n}{3n-4}$.

Inequality (2.9) is relevant since the set $\{|u| \leq s\}$ on its right hand side is a small neighborhood of $\partial\Omega$ (at least if $u > 0$ in Ω) if s is chosen small enough. Thus the $L^{2n/(n-4)}(\Omega)$ bound gets reduced to a question on the regularity of u near $\partial\Omega$.

To prove Theorem 2.8 we take the truncation of $|u|$ at level s as a test function in (1.3) to obtain

$$(2.11) \quad (n-1)s^2 \int_{\{|u|>s\} \cap \{|\nabla u|>0\}} H_u^2 |\nabla u|^2 dx \leq \int_{\{|u|\leq s\}} |\nabla u|^4 dx.$$

Now, (2.9) follows from (2.11) and our geometric Sobolev inequality (2.2) with $p = 2$ and $r = 1$.

When $2 \leq n \leq 3$, from (2.11) and Theorem 2.1 (a), it follows that

$$(2.12) \quad \|u\|_{L^\infty(\Omega)} \leq s + \frac{C(n)}{s} |\Omega|^{\frac{4-n}{2n}} \left(\int_{\{|u|\leq s\}} |\nabla u|^4 dx \right)^{1/2},$$

where $C(n)$ is a constant depending only on n . The *a priori* estimate (2.12) was proved by the first author in [6] in a different way, obtaining the L^∞ estimate also in dimension 4.

The gradient estimate (2.10) follows from the $L^{2n/(n-4)}$ bound with the aid of a technique introduced by Bénilan *et al.* [2] to prove regularity of entropy solutions for p -Laplace equations with L^1 data (see Proposition 4.1 below).

The study of the regularity of semi-stable solutions was motivated by some open problems raised by Brezis and Vázquez [4] about the regularity of extremal solutions. They appear in the following context. Consider positive solutions of

$$(2.13)_\lambda \quad \begin{cases} -\Delta u &= \lambda f(u) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$ is a parameter and f is a C^1 positive increasing function defined on $[0, \infty)$ (in particular $f(0) > 0$) which is superlinear at infinity (*i.e.*, satisfying $f(t)/t \rightarrow +\infty$ as $t \rightarrow +\infty$). Under these assumptions (see the excellent monograph [9] for all these questions), there exists an extremal parameter $\lambda^* \in (0, \infty)$ such that problem $(2.13)_\lambda$ admits a classical minimal solution u_λ for $\lambda \in (0, \lambda^*)$ and admits no weak solution (see Definition 4.4) for $\lambda > \lambda^*$. By minimality it is easy to show that u_λ is a semi-stable solution for $\lambda \in (0, \lambda^*)$. Moreover,

$$u^* := \lim_{\lambda \uparrow \lambda^*} u_\lambda$$

is a weak solution of $(2.13)_{\lambda^*}$, known as the *extremal solution*. Thus, u^* is a semi-stable weak solution of $(2.13)_{\lambda^*}$.

In full generality (*i.e.*, for all domains Ω and all nonlinearities f), the optimal regularity for u^* remains still as open problem. For instance, it is unknown if u^* always lies in the energy class $H_0^1(\Omega)$, or if it is always bounded when $n \leq 9$ (see open problems 1 and 4 in [4]). These questions have a positive answer in the radial case for all nonlinearities (see Remark 2.11 below), and also for general domains and power or exponential type nonlinearities. The optimal L^q and $W^{1,p}$ regularity (depending on the dimension) in the general case is also still unknown.

Nedev [17] proved in the case of convex nonlinearities that $u^* \in L^\infty(\Omega)$ when $n \leq 3$ and $u^* \in L^q(\Omega)$ for all $q < n/(n-4)$ when $n \geq 4$. Note that these regularity results hold for arbitrary smooth domains Ω . In another paper, Nedev [18] also proved that if in addition Ω is strictly convex then $u^* \in H_0^1(\Omega)$. In particular, $u^* \in L^{\frac{2n}{n-2}}(\Omega)$. This is the content of the unpublished preprint [18]. In the present paper, we supply with detailed proofs (slightly modified) of the result in [18]—see Theorem 2.9, Remark 2.10, and subsection 4.3 below.

As in Theorem 2.8, it is also possible to prove that $u^* \in W_0^{1,p}(\Omega)$ for all $p < 4n/(3n-4)$. However, as we said before, the following $W^{1,2} = H^1$ estimate of Nedev [18]—proved using a different argument than ours—is better than the one of Theorem 2.8.

Theorem 2.9 (Nedev [18]). *Let $f : [0, +\infty) \rightarrow \mathbb{R}$ be an increasing positive C^1 function such that $f(t)/t \rightarrow +\infty$ as $t \rightarrow +\infty$. Assume that Ω is a convex smooth bounded domain of \mathbb{R}^n with $n \geq 2$. Then $u^* \in H_0^1(\Omega)$. In particular, $u^* \in L^{\frac{2n}{n-2}}(\Omega)$.*

To prove Theorem 2.9, a Pohožaev identity and the minimality of u_λ is used to obtain

$$\int_{\Omega} |\nabla u_\lambda|^2 \, dx \leq \frac{1}{2} \int_{\partial\Omega} |\nabla u_\lambda|^2 (x \cdot \nu(x)) \, d\sigma \quad \text{for all } \lambda \in (0, \lambda^*),$$

where ν is the outward unit normal to Ω . Then, since Ω is convex, the moving planes method allows to control the right hand side of the previous inequality by $\|u^*\|_{L^1(\Omega)}$. Since u^* is a weak solution of $(2.13)_{\lambda^*}$, and hence $u^* \in L^1(\Omega)$, Theorem 2.9 follows. For the sake of completeness we will prove this result simplifying slightly the original proof of Nedev.

Remark 2.10. Nedev [18] pointed out that Theorem 2.9 also holds for certain non-convex domains. More precisely, let ν be the outward unit normal to Ω and $E := \{x \in \partial\Omega : \text{there exists } \varepsilon > 0 \text{ and a hyperplane } P \text{ such that } P \cap \Omega \cap B_\varepsilon(x) = \{x\}\}$. If there exists $a \in \mathbb{R}^n$ and $\alpha < 0$ such that $(x - a) \cdot \nu(x) \leq \alpha$ for every $x \in \partial\Omega \setminus E$,

then the statement of Theorem 2.9 holds in Ω . Note that this assumption is satisfied by strictly convex domains, annulus, or bean pea shaped domains, for example. See also Remark 4.6 below.

Remark 2.11 (Regularity in the radial case). In [7] it is studied the regularity of semi-stable radially symmetric solutions when the domain is a ball. It is proved that every semi-stable solution, in particular the extremal solution of $(2.13)_\lambda$, is bounded if the dimension $n \leq 9$. For $n \geq 10$, it is proved that such a solution belongs to $W_0^{1,q}(B_1)$ for all $1 \leq q < q_1$, where

$$q_1 := \frac{2n}{n - 2\sqrt{n-1} - 2}.$$

In particular, it belongs to $L^q(B_1)$ for all $1 \leq q < q_0$, where

$$q_0 := \frac{2n}{n - 2\sqrt{n-1} - 4}.$$

It can be shown that these regularity results are sharp by taking explicit semi-stable solutions corresponding to the exponential and power nonlinearities.

Note that the $L^{\frac{2n}{n-4}}(\Omega)$ estimate obtained in Theorem 2.7 differs from the sharp exponent q_0 defined above by the term $2\sqrt{n-1}$.

2.3. Plan of the paper. The paper is organized as follows. In section 3, we prove the geometric-type inequalities stated in Theorem 2.1. In section 4, we deal with semi-stable solutions and we prove the estimates stated in Theorems 2.7 and 2.8. Finally, we prove Theorem 2.9 due to Nedev [18] in an unpublished preprint.

3. GEOMETRIC-TYPE SOBOLEV INEQUALITIES. PROOF OF THEOREM 2.1

The main purpose of this section is to establish Theorem 2.1. Its proof uses two isoperimetric inequalities. The first one is a consequence of the Fleming-Rishel formula [11] and the classical isoperimetric inequality. If $v \in W_0^{1,1}(\Omega)$, then

$$(3.1) \quad A_1 V(t)^{(n-1)/n} \leq P(t) = \frac{d}{dt} \int_{\{|v| \leq t\}} |\nabla v| dx \quad \text{for a.e. } t > 0,$$

where $A_1 := n|B_1|^{1/n}$, $V(t) := |\{x \in \Omega : |v(x)| > t\}|$, and $P(t)$ stands for the perimeter in the sense of De Giorgi, *i.e.*, $P(t)$ is the total variation of the characteristic function of $\{x \in \Omega : |v(x)| > t\}$. A proof of this inequality can be found in [20]. We also note that the distribution function $V(t)$ is differentiable almost everywhere since it is a nonincreasing function.

The second isoperimetric inequality that we use is inequality (2.5), due to Michael and Simon [16] and to Allard [1] —see also Theorem 28.4.1 [5]. We apply it to almost all level sets of $|v|$, where $v \in C_0^\infty(\overline{\Omega})$. We have

$$(3.2) \quad P(t)^{\frac{n-2}{n-1}} \leq A_2 \int_{\{|v|=t\} \cap \{|\nabla v| > 0\}} |H_v| d\sigma \quad \text{for a.e. } t > 0.$$

Here, H_v is the mean curvature of $\{|v| = t\}$ and A_2 is a constant depending only on the dimension $n \geq 2$. Note that, by Sard's theorem, almost every $t \in (0, \|v\|_{L^\infty(\Omega)})$ is a regular value of $|v|$. By definition, if t is a regular value of $|v|$, then $|\nabla v(x)| > 0$ for all $x \in \Omega$ such that $|v(x)| = t$. In particular, if t is a regular value, $S_t := \{x \in \Omega : |v(x)| = t\}$ is a C^∞ immersed $(n-1)$ -dimensional compact hypersurface of \mathbb{R}^n without boundary. Hence, we can apply inequality (2.5) to $S = S_t$ obtaining

(3.2). Note here that, since S could have a finite number of connected components, inequality (2.5) (and (3.2)) for connected manifolds S leads to the same inequality (with same constant) for S with more than one component.

From (3.2) and Jensen inequality, we deduce

$$(3.3) \quad P(t)^{\frac{n-(1+r)}{n-1}} \leq A_2^r \int_{\{|v|=t\} \cap \{|\nabla v|>0\}} |H_v|^r d\sigma \quad \text{for all } r \geq 1.$$

Since we always have $n \geq 1+r$ in Theorem 2.1, we can now use the isoperimetric inequality (3.1) to conclude

$$(3.4) \quad A_1^{\frac{n-(1+r)}{n-1}} V(t)^{\frac{n-(1+r)}{n}} \leq A_2^r \int_{\{|v|=t\} \cap \{|\nabla v|>0\}} |H_v|^r d\sigma \quad \text{for all } r \geq 1.$$

This is the key inequality to prove Theorem 2.1. Note that in the case $r = 0$, inequality (3.4) also holds—it is nothing but the classical isoperimetric inequality (3.1). We start by proving parts (a) and (c).

Proof of Theorem 2.1 (a) and (c). First, we deal with the case $p = 1$ and $r = n-1$. Integrating (3.3) from 0 to $\|v\|_{L^\infty(\Omega)}$ and using the coarea formula, we obtain

$$\|v\|_{L^\infty(\Omega)} \leq A_2^{n-1} \int_{\Omega \cap \{|\nabla v|>0\}} |H_v|^{n-1} |\nabla v| dx,$$

i.e., (2.1) with $C_1 = A_2^{n-1}$.

Assume now $p > 1$ and $r \in [1, n-1]$. Using the coarea formula and that almost every $t \in (0, \|v\|_{L^\infty(\Omega)})$ is a regular value of $|v|$, we have

$$-V'(t) = \int_{\{|v|=t\} \cap \{|\nabla v|>0\}} \frac{d\sigma}{|\nabla v|} \quad \text{for a.e. } t > 0.$$

Hence, by (3.4) and Hölder inequality we obtain

$$A_1^{\frac{n-(1+r)}{n-1}} A_2^{-r} V(t)^{\frac{n-(1+r)}{n}} \leq (-V'(t))^{1/p'} \left(\int_{\{|v|=t\} \cap \{|\nabla v|>0\}} |H_v|^{pr} |\nabla v|^{p-1} d\sigma \right)^{1/p}$$

for a.e. $t > 0$, where $p' = p/(p-1)$, or equivalently,

$$1 \leq A \left(V(t)^{-\frac{n-(1+r)}{n} p'} (-V'(t)) \right)^{1/p'} \left(\int_{\{|v|=t\} \cap \{|\nabla v|>0\}} |H_v|^{pr} |\nabla v|^{p-1} d\sigma \right)^{1/p}$$

for a.e. $t > 0$ such that $V(t) > 0$, where $A = A_1^{-\frac{n-(1+r)}{n-1}} A_2^r$. Integrating the previous inequality with respect to t in $(0, s)$ and using Hölder inequality, we have

$$(3.5) \quad s \leq A \left(\int_{V(s)}^{|\Omega|} \tau^{-\frac{n-(1+r)}{n} p'} d\tau \right)^{1/p'} \left(\int_{\Omega \cap \{|\nabla v|>0\}} |H_v|^{pr} |\nabla v|^p dx \right)^{1/p}$$

for a.e. $s \in (0, \|v\|_{L^\infty(\Omega)})$. Let

$$\beta := -\frac{n-(1+r)}{n} p' + 1 = -\frac{n-p(1+r)}{(p-1)n}.$$

(a) Assume $n < p(1+r)$ and note that $\beta > 0$. Therefore, letting $s \uparrow \|v\|_{L^\infty(\Omega)}$ in (3.5), we obtain

$$\|v\|_{L^\infty(\Omega)} \leq \frac{A|\Omega|^{\frac{p(1+r)-n}{np}}}{\beta^{1/p'}} \left(\int_{\Omega \cap \{|\nabla v| > 0\}} |H_v|^{pr} |\nabla v|^p dx \right)^{1/p},$$

proving the remaining case of assertion (a).

(c) Assume $n = p(1+r)$ and $p > 1$. From (3.5), we obtain

$$s \leq A \left(\int_{V(s)}^{|\Omega|} \frac{d\tau}{\tau} \right)^{1/p'} \left(\int_{\Omega \cap \{|\nabla v| > 0\}} |H_v|^{pr} |\nabla v|^p dx \right)^{1/p} \quad \text{for a.e. } s \in (0, \|v\|_{L^\infty(\Omega)}),$$

and therefore,

$$(3.6) \quad V(s) \leq |\Omega| \exp \left\{ - \left(\frac{s}{AI_p} \right)^{p'} \right\} \quad \text{for a.e. } s \in (0, \|v\|_{L^\infty(\Omega)}),$$

where $I_p := \left(\int_{\Omega \cap \{|\nabla v| > 0\}} |H_v|^{pr} |\nabla v|^p dx \right)^{1/p}$. Let k be any positive integer. Using (3.6) we obtain

$$\begin{aligned} \int_{\Omega} |v|^{kp'} dx &= kp' \int_0^\infty s^{kp'-1} V(s) ds \\ &\leq kp' |\Omega| \int_0^\infty s^{kp'-1} e^{-\left(\frac{s}{AI_p}\right)^{p'}} ds \\ &= k |\Omega| (AI_p)^{kp'} \int_0^\infty \tau^{k-1} e^{-\tau} d\tau \\ &= |\Omega| (AI_p)^{kp'} k!. \end{aligned}$$

Let $C_3 > A$ (remember that here A depends only on n and p since $r = (n-p)/p$) be any positive constant. Then, the previous inequality leads to

$$\int_{\Omega} \exp \left\{ \left(\frac{|v|}{C_3 I_p} \right)^{p'} \right\} dx \leq \sum_{k=0}^{\infty} \left(\frac{A}{C_3} \right)^{kp'} |\Omega| = \frac{C_3^{p'}}{C_3^{p'} - A^{p'}} |\Omega|.$$

This ends the proof of parts (a) and (c) of Theorem 2.1. \square

Remark 3.1. In the previous proof we have obtained the following explicit expressions for the constants in parts (a) and (c) of Theorem 2.1. Here, $A_1 = n|B_1|^{1/n}$ and A_2 denote the constants appearing in (2.4) and (2.5), respectively, which depend only on n .

The constant in the L^∞ estimate of part (a) can be taken to be

$$C_1 = \left(\frac{(p-1)n}{p(1+r)-n} \right)^{1-\frac{1}{p}} A_1^{\frac{1+r-n}{n-1}} A_2^r$$

when $p > 1$, and $C_1 = A_2^{n-1}$ when $p = 1$ and $r = n-1$. Trudinger's type inequality (2.3) holds for all

$$C_3 > A_1^{-\frac{n}{(n-1)p'}} A_2^{\frac{n}{p}-1} =: A$$

and the constant C_4 is given by $C_4 = C_3^{p'}/(C_3^{p'} - A^{p'})$.

Remark 3.2. Assume $p > 1$ and $n > p(1+r)$. Let p_r^* the critical Sobolev exponent defined in Theorem 2.1 (b). Computing the first integral in (3.5), we deduce

$$V(s) \leq |\Omega| \left(\frac{p'}{p_r^*} \left(\frac{|\Omega|^{1/p_r^*}}{AI_p} \right)^{p'} s^{p'} + 1 \right)^{-p_r^*/p'} \quad \text{for a.e. } s \in (0, \|v\|_{L^\infty(\Omega)}),$$

where $I_p := \left(\int_{\Omega \cap \{|\nabla v| > 0\}} |H_v|^{pr} |\nabla v|^p dx \right)^{1/p}$. Noting that

$$\int_{\Omega} |v|^q dx = q \int_0^\infty s^{q-1} V(s) ds,$$

one obtains that, for some constant C depending only on n, p, r , and q ,

$$\left(\int_{\Omega} |v|^q dx \right)^{1/q} \leq C |\Omega|^{\frac{1}{q} - \frac{1}{p_r^*}} \left(\int_{\Omega \cap \{|\nabla v| > 0\}} |H_v|^{pr} |\nabla v|^p dx \right)^{1/p}$$

for all $q < p_r^*$. The constant C may be chosen to be

$$C = \left(\frac{q}{p'} \right)^{\frac{1}{q}} \left(\frac{p'}{p_r^*} \right)^{-\frac{1}{p'}} A \left(\int_0^\infty \tau^{\frac{q}{p'} - 1} (\tau + 1)^{-\frac{p_r^*}{p'}} d\tau \right)^{1/q},$$

which is finite if and only if $q < p_r^*$. However, using this argument it is not possible to obtain the inequality with the critical Sobolev exponent $q = p_r^*$. Although we could introduce Schwarz (or decreasing) symmetrization in order to get the critical exponent p_r^* , we use the following slightly different argument.

Now, we prove Theorem 2.1 (b).

Proof of Theorem 2.1 (b). Assume $n > p(1+r)$ and let $p_r^* = np/(n - p(1+r))$. Integrating (3.4) from 0 to $M := \|v\|_{L^\infty(\Omega)}$, we obtain

$$(3.7) \quad A_1^{\frac{n-(1+r)}{n-1}} \int_0^M V(t)^{\frac{n-(1+r)}{n}} dt \leq A_2^r \int_{\Omega \cap \{|\nabla v| > 0\}} |H_v|^r |\nabla v| dx \quad \text{for all } r \geq 1.$$

Let

$$W(t) := \left(\int_0^t V(s)^{\frac{n-(1+r)}{n}} ds \right)^{1_r^*}.$$

Using that $V(t)$ is a nonincreasing function, we easily deduce

$$1_r^* t^{1_r^*-1} V(t) \leq W'(t) \quad \text{for a.e. } t \in (0, M).$$

Hence, integrating from 0 to M , we get

$$\int_{\Omega} |v|^{1_r^*} dx = 1_r^* \int_0^M t^{1_r^*-1} V(t) dt \leq W(M) = \left(\int_0^M V(t)^{\frac{n-(1+r)}{n}} dt \right)^{1_r^*}.$$

Combining this with (3.7) we obtain

$$(3.8) \quad A_1^{\frac{n-(1+r)}{n-1}} \left(\int_{\Omega} |v|^{1_r^*} dx \right)^{1/1_r^*} \leq A_2^r \int_{\Omega \cap \{|\nabla v| > 0\}} |H_v|^r |\nabla v| dx,$$

i.e., assertion (b) for $p = 1$.

For $p > 1$, we only need to apply inequality (3.8) with $|v|$ replaced by $|v|^\gamma$ and $\gamma = p_r^*/1_r^*$ (noting that the level sets of $|v|$ and $|v|^\gamma$ are the same, and hence, their mean curvatures coincide) and use Hölder inequality to conclude (2.2). \square

Remark 3.3. Inequality (2.2) in Theorem 2.1 (b) holds with the constant

$$C_2 = \frac{n - (1 + r)}{n - p(1 + r)} p \left(n |B_1|^{1/n} \right)^{-\frac{n - (1 + r)}{n - 1}} A_2^r,$$

where A_2 is the constant appearing in (2.5).

Remark 3.4. Here we show that the inequalities in Theorem 2.1 (both the Sobolev and the Morrey inequalities) do not hold when $r \in (0, 2p^{-1} - 1)$ and $1 \leq p < 2$. In particular, they do not hold for $r \in (0, 1)$ and $p = 1$.

We also study the geometric inequalities behind (3.3) and (3.4). That is, we study the inequalities

$$(3.9) \quad |\partial\Omega|^{\frac{n - (1 + pr)}{n - 1}} \leq C \int_{\partial\Omega} |H|^{pr} d\sigma$$

and

$$(3.10) \quad |\Omega|^{\frac{n - (1 + pr)}{n}} \leq C \int_{\partial\Omega} |H|^{pr} d\sigma,$$

where H is the mean curvature of $\partial\Omega \subset \mathbb{R}^n$. We show that for every constant $C = C(n, p, r)$ inequalities (3.9) and (3.10) fail, even among convex sets $\Omega \subset \mathbb{R}^n$, when $r \in (0, 1/p)$ and $p \geq 1$.

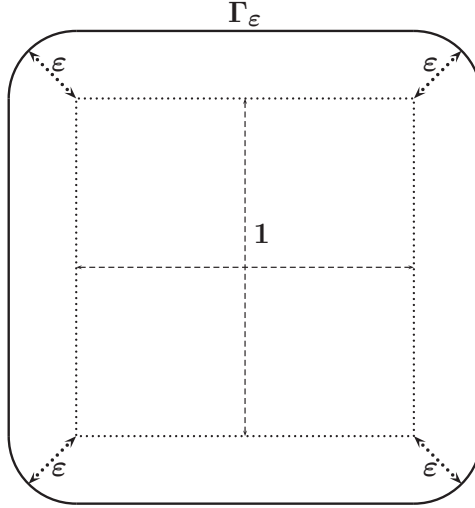


FIGURE 1. Level sets of v .

To see all this, let $Q_1 = (0, 1)^n$ be the open unitary cube of \mathbb{R}^n , $n \geq 2$. Given $\varepsilon \in (0, 1/2)$, set $\Gamma_\varepsilon := \{x \in \mathbb{R}^n \setminus Q_1 : \text{dist}(x, Q_1) = \varepsilon\}$ and Ω_ε to be its bounded interior. Let H_{Γ_ε} be the mean curvature of Γ_ε and $A_\varepsilon := \{x \in \Gamma_\varepsilon : H_{\Gamma_\varepsilon}(x) \neq 0\}$. Note that

$$(3.11) \quad H_{\Gamma_\varepsilon} \equiv 0 \text{ on } \Gamma_\varepsilon \setminus A_\varepsilon, \quad |A_\varepsilon| \leq c_1 \varepsilon, \quad \text{and} \quad |H_{\Gamma_\varepsilon}| \leq c_2 \varepsilon^{-1} \text{ on } A_\varepsilon,$$

where c_1 and c_2 are constants depending only on n . Therefore, since $r > 0$,

$$(3.12) \quad \int_{\Gamma_\varepsilon} |H_{\Gamma_\varepsilon}|^{pr} d\sigma \leq c_3 \varepsilon^{1 - pr},$$

where c_3 is a constant depending only on n , p , and r . Since $|\Gamma_\varepsilon| > 1$ and $|\Omega_\varepsilon| > 1$ for all $\varepsilon \in (0, 1/2)$, and the right hand side of (3.12) tends to zero, as ε goes to zero, when $r \in (0, 1/p)$, we obtain that (3.9) and (3.10) do not hold for $r \in (0, 1/p)$, as claimed.

Although Γ_ε is not C^∞ (since $\text{dist}(\cdot, Q_1)$ is not a C^∞ function), the same facts hold for C^∞ immersed $(n-1)$ -dimensional compact hypersurfaces of \mathbb{R}^n . Indeed, there exists $\tilde{d} \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leq \tilde{d} \leq 1$, $\tilde{d} \equiv 0$ in Q_1 , $\tilde{d} \equiv 1$ in $\{x \in \mathbb{R}^n : \text{dist}(x, Q_1) \geq 1\}$, $|\nabla \tilde{d}| \leq 2$, and its level sets $\tilde{\Gamma}_\varepsilon := \{x \in \mathbb{R}^n : \tilde{d}(x) = \varepsilon\}$, $0 < \varepsilon < 1$, satisfy (3.11) (and hence (3.12)). This can be seen choosing a hypersurface $\tilde{\Gamma}_1$ coinciding with Γ_1 in its flat parts and smoothing it in vertex, edges, etc. Then we define $\tilde{\Gamma}_\varepsilon$ for $0 < \varepsilon < 1$ as an homotethy with respect to the vertex, edges, etc., of the cube. In this way, $\tilde{\Gamma}_\varepsilon$ produces a foliation of $\{x \in \mathbb{R}^n : 0 < \text{dist}(x, Q_1) < 1\}$. We finally define $\tilde{d}(x) = \varepsilon$ if and only if $x \in \tilde{\Gamma}_\varepsilon$.

Now, we can prove that the inequalities in Theorem 2.1 fail whenever $r \in (0, 2p^{-1} - 1)$. Let v be a positive function whose level sets are $\tilde{\Gamma}_\varepsilon$. More precisely, let $\psi : [0, +\infty) \rightarrow \mathbb{R}$ be any decreasing C^∞ function such that $\psi(s) = 0$ for $s \geq 1$ and $\psi^{(i)}(0) = 0$ for all $i \geq 1$. Given $\varepsilon_0 \in (0, 1)$, we define $v(x) = \psi(\tilde{d}(x)/\varepsilon_0)$ in $\mathbb{R}^n \setminus [0, 1]^n$ and $v = \psi(0)$ in $[0, 1]^n$. Note that $v \in C^\infty(\overline{\Omega})$ where Ω is the bounded interior of $\tilde{\Gamma}_{\varepsilon_0}$.

Using the coarea formula, $|\nabla \tilde{d}| \leq 2$, (3.12), and the change of variables $v = \psi(\tilde{d}/\varepsilon_0) = t = \psi(s)$, it is easy to check that

$$\begin{aligned} \int_{\Omega \cap \{|\nabla v| > 0\}} |H_v|^{pr} |\nabla v|^p dx &= \int_0^{\psi(0)} \int_{\{v=t\} \cap \{|\nabla v| > 0\}} |H_v|^{pr} |\nabla v|^{p-1} d\sigma dt \\ &\leq C \int_0^1 |\psi'(s)|^p \varepsilon_0^{-(p-1)} \int_{\tilde{\Gamma}_{\varepsilon_0 s}} |H_{\varepsilon_0 s}|^{pr} d\sigma ds \\ &\leq \frac{C}{\varepsilon_0^{p-1}} \varepsilon_0^{1-pr} \int_0^1 |\psi'(s)|^p s^{1-pr} ds, \end{aligned}$$

where C is a constant depending only on n , p , and r . Note that the right hand side of this inequality tends to zero as ε_0 goes to 0 if $r < 2p^{-1} - 1$. On the other hand, it is clear that, for any $q \geq 1$,

$$\|v\|_{L^q(\Omega)} \geq \left(\int_{Q_1} |v|^q dx \right)^{1/q} = \psi(0) = \|v\|_{L^\infty(\Omega)} > 0.$$

Therefore, a necessary condition in order that Theorem 2.1 holds (in the range $r > 0$) is $r \geq 2p^{-1} - 1$. In particular, if $p = 1$ the necessary condition is that $r \geq 1$.

Remark 3.5. We derive two more inequalities involving the perimeter $P(t)$ of the level sets. On the one hand, using (3.3), integrating with respect to t in $(0, \|v\|_{L^\infty(\Omega)})$, and using the coarea formula, we obtain

$$\|v\|_{L^\infty(\Omega)} \leq A_2^r \int_{\Omega \cap \{|\nabla v| > 0\}} P(v)^{\frac{1+r-n}{n-1}} |H_v|^r |\nabla v| dx \quad \text{for all } n \geq 2, r \geq 1.$$

On the other hand, note that the total variation of v may be written as

$$\int_{\Omega} |\nabla v| dx = \int_0^{\|v\|_{L^\infty(\Omega)}} P(t) dt,$$

and that by (3.3) we have

$$1 \leq A_2^{n-1} P(t)^{\frac{1+r-n}{r}} \left(\int_{\{|v|=t\} \cap \{|\nabla v|>0\}} |H_v|^r d\sigma \right)^{\frac{n-1}{r}}.$$

In the case $2 \leq n < 1+r$ (which is not considered in Theorem 2.1), integrating the previous inequality with respect to t in $(0, \|v\|_{L^\infty(\Omega)})$ and using Hölder inequality, we obtain

$$\|v\|_{L^\infty(\Omega)} \leq A_2^{n-1} \left(\int_{\Omega} |\nabla v| dx \right)^{\frac{1+r-n}{r}} \left(\int_{\Omega \cap \{|\nabla v|>0\}} |H_v|^r |\nabla v| dx \right)^{\frac{n-1}{r}}.$$

4. SEMI-STABLE SOLUTIONS. PROOF OF THEOREMS 2.8, 2.7, AND 2.9

This section deals with semi-stable solutions. We apply Theorem 2.1 to prove Theorems 2.8 and 2.7. Finally we prove Theorem 2.9 using a Pohožaev identity and the fact that the extremal solution u^* is the increasing limit in L^1 of minimal classical solutions.

To obtain the $L^{\frac{2n}{n-4}}$ estimate of Theorems 2.8 and 2.7 we use the semi-stability condition (1.2) with test function $\xi = |\nabla u|\eta$, where u is a smooth semi-stable solution of (1.1) and η vanishes on $\partial\Omega$ and is still arbitrary. With this choice one has

$$\begin{aligned} \int_{\Omega \cap \{|\nabla u|>0\}} |B_u|^2 |\nabla u|^2 \eta^2 dx &\leq \int_{\Omega \cap \{|\nabla u|>0\}} (|\nabla_T |\nabla u||^2 + |B_u|^2 |\nabla u|^2) \eta^2 dx \\ (4.1) \qquad \qquad \qquad &\leq \int_{\Omega} |\nabla u|^2 |\nabla \eta|^2 dx, \end{aligned}$$

for every Lipschitz function η in $\bar{\Omega}$ with $\eta|_{\partial\Omega} \equiv 0$ (see for instance Proposition 2.2 of [6] and references therein). Here, ∇_T denotes the tangential gradient along a level set of $|u|$ and

$$|B_u(x)|^2 = \sum_{i=1}^{n-1} \kappa_i^2(x),$$

where $\kappa_i(x)$ are the principal curvatures of the level set of $|u|$ passing through x , for a given $x \in \Omega \cap \{|\nabla u|>0\}$. Now, noting that $(n-1)H_u^2 \leq |B_u|^2$, we deduce inequality (1.3) from (4.1):

$$(4.2) \qquad (n-1) \int_{\Omega \cap \{|\nabla u|>0\}} H_u^2 |\nabla u|^2 \eta^2 dx \leq \int_{\Omega} |\nabla u|^2 |\nabla \eta|^2 dx.$$

4.1. Proof of Theorem 2.8. The $L^{\frac{2n}{n-4}}$ estimate will follow from (4.2). Instead, the $W^{1,p}$ estimates of Theorem 2.8 will use the following result. It holds for solutions of the linear problem

$$(4.3) \qquad \begin{cases} -\Delta u &= h(x) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{cases}$$

Proposition 4.1. *Assume $n \geq 3$ and $h \in L^1(\Omega)$. If $u \in W_0^{1,1}(\Omega) \cap L^q(\Omega)$ is a solution (in the distributional sense) of (4.3) for some $q \geq n/(n-2)$, then*

$$\int_{\Omega} |\nabla u|^p dx \leq p|\Omega| + \left(\frac{p_q}{p} - 1 \right)^{-1} \{ \|u\|_{L^q(\Omega)}^q + \|h\|_{L^1(\Omega)} \}$$

for all $p < p_q := \frac{2q}{q+1}$.

Remark 4.2. Assume $h \in L^1(\Omega)$. By standard estimates for elliptic equations, there exists a constant C depending only on n , p , and $|\Omega|$, such that

$$\int_{\Omega} |\nabla u|^p dx \leq C \|h\|_{L^1(\Omega)} \quad \text{for all } p < \frac{n}{n-1}$$

for every solution u of (4.3). The critical exponent $p = n/(n-1)$ can not be reached. In Proposition 4.1, under the additional assumption $u \in L^q(\Omega)$ for some $q \geq n/(n-2)$, we improve the previous estimate; note that $p_q := 2q/(q+1) \geq n/(n-1)$.

The exponent p_q in Proposition 4.1 is the same as the one in the Gagliardo-Nirenberg interpolation inequality

$$\|\nabla u\|_{L^{p_q}(\Omega)} \leq C \|u\|_{W^{2,1}(\Omega)}^{1/2} \|u\|_{L^q(\Omega)}^{1/2}.$$

Note that in Proposition 4.1 we assume $-\Delta u = h \in L^1(\Omega)$ and $u \in L^q(\Omega)$.

The proof of Proposition 4.1 is based in a technique introduced by B nilan *et al.* [2] to obtain gradient estimates for the entropy solution of problem (4.3) with the Laplacian replaced by the p -Laplacian.

Proof of Proposition 4.1. Multiplying (4.3) by $T_s u = \max\{-s, \min\{s, u\}\}$ we obtain

$$\int_{\{|u| \leq s\}} |\nabla u|^2 dx = \int_{\Omega} h(x) T_s u dx \leq s \|h\|_{L^1(\Omega)}.$$

From this, we deduce

$$\begin{aligned} s^q |\{| \nabla u | > s^{(q+1)/2} \}| &\leq s^q \int_{\{| \nabla u | > s^{(q+1)/2} \} \cap \{|u| \leq s\}} \left(\frac{| \nabla u |}{s^{(q+1)/2}} \right)^2 dx + s^q \int_{\{|u| > s\}} dx \\ &\leq \|h\|_{L^1(\Omega)} + s^q V(s), \quad \text{for a.e. } s > 0. \end{aligned}$$

Recall that $V(s) = |\{x \in \Omega : |u(x)| > s\}|$. Letting $t = s^{(q+1)/2}$, we have

$$(4.4) \quad t^{2q/(q+1)} |\{| \nabla u | > t \}| \leq \sup_{\sigma > 0} \left\{ \sigma^q V(\sigma) \right\} + \|h\|_{L^1(\Omega)}, \quad \text{for a.e. } t > 0.$$

Moreover, since

$$\sigma^q V(\sigma) \leq \sigma^q \int_{\{|u| > \sigma\}} \frac{|u|^q}{\sigma^q} dx \leq \int_{\Omega} |u|^q dx = \|u\|_{L^q(\Omega)}^q, \quad \text{for a.e. } \sigma > 0,$$

we have $\sup_{\sigma > 0} \left\{ \sigma^q V(\sigma) \right\} \leq \|u\|_{L^q(\Omega)}^q$. Therefore, from (4.4) we deduce

$$\begin{aligned} \int_{\Omega} |\nabla u|^p dx &= p \int_0^\infty t^{p-1} |\{| \nabla u | > t \}| dt \\ &\leq p |\Omega| + p \int_1^\infty t^{p-1} t^{-\frac{2q}{q+1}} \left(\|u\|_{L^q(\Omega)}^q + \|h\|_{L^1(\Omega)} \right) dt, \end{aligned}$$

proving the proposition. \square

Using (4.2), Proposition 4.1, and applying Theorem 2.1 (b) to $|u| - s$ with Ω replaced by $\{x \in \Omega : |u(x)| > s\}$, we prove Theorem 2.8.

Proof of Theorem 2.8. Since $g \in C^\infty$, we have that $u \in C^\infty(\bar{\Omega})$. Recall that we assume $n \geq 5$. By taking $\eta = T_s u = \max\{-s, \min\{s, u\}\}$ in (4.2), we obtain

$$(4.5) \quad (n-1) \int_{\{|u|>s\} \cap \{|\nabla u|>0\}} H_u^2 |\nabla u|^2 dx \leq \frac{1}{s^2} \int_{\{|u|\leq s\}} |\nabla u|^4 dx,$$

for all $s > 0$. We apply Theorem 2.1 (b) to $v = u - s \in C^\infty(\bar{\Omega})$ with $p = 2$ and $r = 1$, replacing Ω by each component of $\{x \in \Omega : u(x) > s\}$ (which is C^∞ for a.e. s). Using also (4.5) we deduce

$$\begin{aligned} \left(\int_{\{u>s\}} (u-s)^{\frac{2n}{n-4}} dx \right)^{\frac{n-4}{2n}} &\leq C_2 \left(\int_{\{u>s\} \cap \{|\nabla u|>0\}} H_u^2 |\nabla u|^2 dx \right)^{\frac{1}{2}} \\ &\leq \frac{C(n)}{s} \left(\int_{\{u\leq s\}} |\nabla u|^4 dx \right)^{\frac{1}{2}}, \end{aligned}$$

for a.e. $s > 0$, where $C(n)$ depends only on n . Doing the same argument for $-u - s$ in $\{-u > s\}$ we conclude (2.9).

Finally, (2.10) follows applying Proposition 4.1 with $q = 2n/(n-4)$. \square

4.2. Proof of Theorem 2.7. To prove Theorem 2.7 we need to control the right hand side of (2.9). We accomplish this using a boundary regularity result for positive solutions in convex domains. More precisely, we use the following result from [12, 10] (see also [9] for its proof).

Proposition 4.3 ([12, 10]). *Let f be any locally Lipschitz function and let Ω be a smooth bounded domain of \mathbb{R}^n . Let u be any positive classical solution of (1.1).*

If Ω is convex, then there exist positive constants ε and γ depending only on the domain Ω such that for every $x \in \Omega$ with $\text{dist}(x, \partial\Omega) < \varepsilon$, there exists a set $I_x \subset \Omega$ with the following properties:

$$|I_x| \geq \gamma \quad \text{and} \quad u(x) \leq u(y) \quad \text{for all } y \in I_x.$$

As a consequence,

$$\|u\|_{L^\infty(\Omega_\varepsilon)} \leq \frac{1}{\gamma} \|u\|_{L^1(\Omega)}, \quad \text{where } \Omega_\varepsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) < \varepsilon\}.$$

We recall (see [9]) that it is well known that the extremal solution u^* belongs to $L^1(\Omega)$ and it is a weak solution of (2.13) $_{\lambda^*}$ in the following sense.

Definition 4.4. Let $\delta(x) := \text{dist}(x, \partial\Omega)$. We say that $u \in L^1(\Omega)$ is a *weak solution* of (1.1) if $g(u)\delta \in L^1(\Omega)$ and

$$\int_{\Omega} u(-\Delta\varphi) dx = \int_{\Omega} g(u)\varphi dx \quad \text{for all } \varphi \in C^2(\bar{\Omega}) \text{ with } \varphi|_{\partial\Omega} = 0.$$

Since $u^* \in L^1(\Omega)$, from Proposition 4.3 we deduce next that u^* is bounded (and smooth) in a neighborhood of the boundary if the domain is convex. This and Theorem 2.8 give Theorem 2.7.

Proof of Theorem 2.7. Assume first that $f \in C^\infty(\mathbb{R})$. Let $u_\lambda \in C^\infty(\bar{\Omega})$ be the minimal solution of (2.13) $_\lambda$ for $\lambda \in (0, \lambda^*)$. By Proposition 4.3, and noting that the extremal solution u^* is the increasing limit of $\{u_\lambda\}$, there exist constants ε and γ independent of λ such that

$$(4.6) \quad \|u_\lambda\|_{L^\infty(\Omega_\varepsilon)} \leq \frac{1}{\gamma} \|u^*\|_{L^1(\Omega)} \quad \text{for all } \lambda < \lambda^*,$$

where

$$\Omega_\varepsilon := \{x \in \Omega : \text{dist}(x, \partial\Omega) < \varepsilon\}.$$

By taking ε smaller if necessary, we may assume that Ω_δ is C^∞ for every $0 < \delta \leq \varepsilon$.

We can conclude the proof in two ways. First, we proceed as in the proof of Proposition 3.1 in [6]. For this, note that if $\lambda^*/2 < \lambda < \lambda^*$, then

$$u_\lambda \geq u_{\lambda^*/2} > c \text{dist}(\cdot, \partial\Omega)$$

for some positive constant c independent of $\lambda \in (\lambda^*/2, \lambda^*)$. Therefore, letting

$$\tilde{s} := c \frac{\varepsilon}{2},$$

we have

$$\{x \in \Omega : u_\lambda(x) \leq \tilde{s}\} \subset \Omega_{\varepsilon/2}.$$

We now use (2.9) in Theorem 2.8 with s replaced by \tilde{s} . It suffices to bound $\|u_\lambda\|_{W^{1,4}(\Omega_{\varepsilon/2})}$. But u_λ is a solution of the linear equation $-\Delta u_\lambda = h(x) := \lambda f(u_\lambda(x))$ in Ω_ε and $u_\lambda = 0$ on $\partial\Omega$ (which is one part of $\partial\Omega_\varepsilon$). On the other hand, $\partial\Omega \cup \Omega_{\varepsilon/2}$ has compact closure contained in $\partial\Omega \cup \Omega_\varepsilon$, and both sets are C^∞ . By (4.6), both u_λ and the right hand side h are bounded independently of λ . Hence, by interior and boundary estimates for the linear Poisson equation, we deduce a bound for $\|u_\lambda\|_{W^{1,4}(\Omega_{\varepsilon/2})}$ independent of λ . Letting λ tend to λ^* , we obtain $u^* \in L^{\frac{2n}{n-4}}(\Omega)$.

Our second proof is perhaps more direct; it does not use regularity for the linear problem. Here we choose a regular value s of u (and thus $\{x \in \Omega : u_\lambda(x) > s\}$ is smooth) such that

$$\frac{1}{\gamma} \|u^*\|_{L^1(\Omega)} \leq s \leq \frac{2}{\gamma} \|u^*\|_{L^1(\Omega)}.$$

By (4.6) we have

$$(4.7) \quad \Omega_\varepsilon \subset \{x \in \Omega : u_\lambda(x) \leq s\}.$$

Now, we use

$$\eta(x) = \begin{cases} \text{dist}(x, \partial\Omega) & \text{in } \Omega_\varepsilon = \{\text{dist}(x, \partial\Omega) < \varepsilon\}, \\ \varepsilon & \text{in } \{\text{dist}(x, \partial\Omega) \geq \varepsilon\} \end{cases}$$

as a test function in (4.2). Using (4.7) we obtain

$$(n-1)\varepsilon^2 \int_{\{u_\lambda > s\} \cap \{|\nabla u_\lambda| > 0\}} H_{u_\lambda}^2 |\nabla u_\lambda|^2 dx \leq \int_{\{u_\lambda < s\}} |\nabla u_\lambda|^2 dx.$$

Multiplying equation (2.13) $_\lambda$ by $T_s u_\lambda = \min\{s, u_\lambda\}$ we have

$$\int_{\{u_\lambda < s\}} |\nabla u_\lambda|^2 dx = \lambda \int_{\Omega} f(u_\lambda) T_s u_\lambda dx \leq \lambda^* s \|f(u^*)\|_{L^1(\Omega)}.$$

Note that $\|f(u^*)\|_{L^1(\Omega)} < \infty$ since it is well known that $f(u^*) \text{dist}(\cdot, \partial\Omega) \in L^1(\Omega)$ in general smooth domains and if in addition Ω is convex then u^* , and thus $f(u^*)$, are bounded in Ω_ε by (4.6).

Therefore, using Theorem 2.1 (b) applied to $v = u_\lambda - s$, with $p = 2$ and $r = 1$, and replacing Ω by each component of $\{x \in \Omega : u_\lambda(x) > s\}$ (which is smooth), we deduce

$$\left(\int_{\{u_\lambda > s\}} (u_\lambda - s)^{\frac{2n}{n-4}} dx \right)^{\frac{n-4}{2n}} \leq \frac{C_2}{\varepsilon \sqrt{n-1}} (\lambda^* s \|f(u^*)\|_{L^1(\Omega)})^{\frac{1}{2}}$$

for all $\lambda \in (0, \lambda^*)$. In particular, letting λ tend to λ^* , we obtain $u^* \in L^{\frac{2n}{n-4}}(\Omega)$.

In case that f is only $C^1(\mathbb{R})$ then one can make an easy approximation argument to obtain the same result (see proof of Theorem 1.2 in [6] for the details). \square

Remark 4.5. As a consequence of Theorem 2.8, if $u \in L^1(\Omega)$ is a weak solution of (1.1) (in the sense of Definition 4.4) which is bounded in a neighborhood of $\partial\Omega$ and which is the $L^1(\Omega)$ limit of a sequence of classical semi-stable solutions of (1.1), then $u \in L^{2n/(n-4)}(\Omega)$ and $u \in W_0^{1,p}(\Omega)$ for all $p < 4n/(3n-4)$. In particular,

$$u \in L^2(\Omega) \cap W_0^{1,4/3}(\Omega)$$

independently of the dimension n . For general solutions (not necessarily semi-stable) the best regularity that one expects assuming only $g(u) \in L^1(\Omega)$ is $u \in L^q(\Omega) \cap W_0^{1,p}(\Omega)$ for all $1 \leq q < n/(n-2)$ and $1 \leq p < n/(n-1)$. Hence, semi-stable solutions enjoy more regularity than general solutions.

4.3. Proof of Theorem 2.9. In an unpublished paper, Nedev [18] proved that the extremal solution u^* lies in the energy class H_0^1 , independently of the dimension, when Ω is strictly convex. For this, he used a Pohožaev identity, an upper bound independent of λ for the energy of the minimal solutions u_λ , and the fact that u^* is bounded (and hence regular) in a neighborhood of the boundary. Here, for the sake of completeness, we give a proof of Nedev's result.

Recall that the energy functional associated to $(2.13)_\lambda$ is given by

$$J_\lambda(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega F(u) dx, \quad F(u) := \int_0^u f(s) ds.$$

In [18] an upper bound of $J_\lambda(u_\lambda)$ is proved by using the parabolic equation associated to $(2.13)_\lambda$, $u_t - \Delta u = \lambda f(u)$. This equation was studied by Brezis *et al.* [3]. The proof that we present here uses a different, purely elliptic, argument at this point.

Proof of Theorem 2.9. Let u_λ be the minimal solution of $(2.13)_\lambda$ and let ν be the outward unit normal to Ω . Multiplying $(2.13)_\lambda$ by $x \cdot \nabla u_\lambda$ it is standard to obtain the following Pohožaev identity:

$$(4.8) \quad \int_\Omega |\nabla u_\lambda|^2 dx = \frac{1}{2} \int_{\partial\Omega} |\nabla u_\lambda|^2 x \cdot \nu(x) d\sigma + n J_\lambda(u_\lambda)$$

for all $\lambda \in (0, \lambda^*)$. Since the minimal solution u_λ is the only solution of $(2.13)_\lambda$ in $\{u \in H_0^1(\Omega) : 0 \leq u \leq u_\lambda\}$, it is also the absolute minimizer of J_λ in this convex set. Hence, we have $J_\lambda(u_\lambda) \leq J_\lambda(0) = 0$ for every $\lambda \in (0, \lambda^*)$.

Therefore, from (4.8) one deduces that

$$(4.9) \quad \int_\Omega |\nabla u_\lambda|^2 dx \leq \frac{1}{2} \int_{\partial\Omega} |\nabla u_\lambda|^2 x \cdot \nu(x) d\sigma, \quad \text{for all } \lambda \in (0, \lambda^*).$$

Now, since Ω is convex, there exist positive constants ε and γ depending only on the domain Ω such that (4.6) holds. As a consequence, $\|f(u_\lambda)\|_{L^\infty(\Omega_\varepsilon)} \leq \|f\|_{L^\infty(0,\alpha)}$ for all $\lambda \in (0, \lambda^*)$, where α is a constant depending only on Ω and $\|u^*\|_{L^1(\Omega)}$ —and thus independent of λ . By (4.6), also u_λ is bounded in Ω_ε independently of λ . Hence, using boundary estimates at $\partial\Omega$ for the linear Poisson equation $-\Delta u_\lambda = \lambda f(u_\lambda(x))$ in Ω_ε , we deduce a bound for the right hand side of inequality (4.9) independent of λ . Making λ tend to λ^* we conclude the proof. \square

Remark 4.6. As mentioned in [18], Theorem 2.9 holds for some nonconvex domains such as annulus or bean pea shaped domains. Indeed, using Pohožaev identity (obtained multiplying (2.13) $_{\lambda}$ by $(x - a) \cdot \nabla u$) and the fact that $J_{\lambda}(u_{\lambda}) \leq 0$, one obtains

$$(4.10) \quad \int_{\Omega} |\nabla u_{\lambda}|^2 dx \leq \frac{1}{2} \int_{\partial\Omega} |\nabla u_{\lambda}|^2 (x - a) \cdot \nu(x) d\sigma \quad \text{for all } \lambda \in (0, \lambda^*).$$

Let $E := \{x \in \partial\Omega : \text{there exists } \varepsilon > 0 \text{ and a hyperplane } P \text{ such that } P \cap \bar{\Omega} \cap B_{\varepsilon}(x) = \{x\}\}$. By using the moving planes method, as in the proof of Proposition 4.3, it can be seen that u^* is bounded (by a constant independent of λ) and regular in a neighborhood in Ω of any compact subset of E . In particular, if there exists $a \in \mathbb{R}^n$ and $\alpha < 0$ such that $(x - a) \cdot \nu(x) \leq \alpha$ for every $x \in \partial\Omega \setminus E$ one obtains from (4.10) that $u^* \in H_0^1(\Omega)$.

REFERENCES

1. W.K. Allard, *On the first variation of a varifold*, Ann. Math. **95** (1972), 417–491.
2. Ph. B  nilan, L. Boccardo, T. Gallou  t, R. Gariepy, M. Pierre, and J.L. V  zquez, *An L^1 -theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Ann. Sc. Norm. Super. Pisa, Cl. Sci. **22** (1995), 241–273.
3. H. Brezis, T. Cazenave, Y. Martel, and A. Ramiandrisoa, *Blow-up for $u_t - \Delta u = g(u)$ revisited*, Adv. Differential Equations **1** (1996), 73–90.
4. H. Brezis and J.L. V  zquez, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Compl. Madrid **10** (1997), 443–469.
5. Yu.D. Burago and V.A. Zalgaller, *Geometric inequalities*, Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1988.
6. X. Cabr  , *Regularity of minimizers of semilinear elliptic problems up to dimension 4*, Comm. Pure Appl. Math. **63** (2010), 1362–1380.
7. X. Cabr   and A. Capella, *Regularity of radial minimizers and extremal solutions of semilinear elliptic equations*, J. Funct. Anal. **238** (2006), 709–733.
8. L.A. Caffarelli, R. Kohn, and L. Nirenberg, *First order interpolation inequalities with weights*, Compositio Math. **53** (1984), 259–275.
9. L. Dupaigne, *Stable solutions to elliptic partial differential equations*, Monographs and Surveys in Pure and Applied Mathematics, 2011.
10. D.G. de Figueiredo, P.-L. Lions, and R.D. Nussbaum, *A priori estimates and existence of positive solutions of semilinear elliptic equations*, J. Math. Pures Appl. **61** (1982), 41–63.
11. W.H. Fleming and R. Rishel, *An integral formula for total gradient variation*, Arch. Math. **11** (1960), 218–222.
12. B. Gidas, W.M. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.
13. P. Guan and J. Li, *The quermassintegral inequalities for starshaped domains*, Adv. Math. **221** (2009), 1725–1732.
14. P. Guan, X.-N. Ma, N. Trudinger, and X. Zhu, *A form of Alexandrov-Fenchel inequality*, Pure Appl. Math. Q. **6** (2010), 999–1012.
15. A. Kufner and L.-E. Persson, *Weighted inequalities of Hardy type*, World Scientific Publishing Co., Inc., River Edge, NJ, 2003.
16. J.H. Michael and L. Simon, *Sobolev and mean value inequalities on generalized submanifolds of \mathbb{R}^n* , Comm. Pure Appl. Math. **26** (1973), 361–379.
17. G. Nedev, *Regularity of the extremal solution of semilinear elliptic equations*, C. R. Acad. Sci. Paris S  r. I Math. **330** (2000), 997–1002.
18. G. Nedev, *Extremal solution of semilinear elliptic equations*, Preprint 2001.
19. F. Schulze, *Nonlinear evolution by mean curvature and isoperimetric inequalities*, J. Differential Geometry **79** (2008), 197–241.
20. G. Talenti, *Nonlinear elliptic equations, rearrangements of functions and Orlicz spaces*, Ann. Mat. Pura Appl. **120** (1979), 160–184.

21. N.S. Trudinger, *On new isoperimetric inequalities and symmetrization*, J. reine angew. Math. **488** (1997), 203–220.

ICREA AND UNIVERSITAT POLITÈCNICA DE CATALUNYA, DEPARTAMENT DE MATEMÀTICA APLICADA I, DIAGONAL 647, 08028 BARCELONA, SPAIN
E-mail address: `xavier.cabre@upc.edu`

UNIVERSITAT DE BARCELONA, DEPARTAMENT DE MATEMÀTICA APLICADA I ANÀLISI, GRAN VIA 585, 08007 BARCELONA, SPAIN
E-mail address: `msanchon@maia.ub.es`